TRANSCENDENCE OF ZEROS OF MODULAR FORMS

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Abstract. In this paper, we show that for several different families of modular forms, the zeros of these modular forms are transcendental (aside from a finite number of exceptions). These families in particular include the Eisenstein series for several different $\Gamma_0^+(p)$ and $\Gamma_0(p)$, cuspidal projections of products of Eisenstein series, and certain forms in the Miller basis. Related results have been previously shown before; we put these previous results into a more general framework for showing transcendence of zeros.

1. Introduction

In this paper, we investigate when the zeros of modular forms are transcendental. Let $\mathcal{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ denote the complex upper half-plane. And recall that $\text{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ with fundamental domain

$$\mathcal{F} := \left\{ z = x + iy \in \mathcal{H} : |z| \geq 1, -\frac{1}{2} \leq x \leq 0 \right\} \cup \left\{ z \in \mathcal{H} : |z| > 1, 0 < x < \frac{1}{2} \right\}.$$

For even integers $k \geq 4$, let $E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz + d)^{-k}$ denote the normalized Eisenstein series of weight k and full modular group $\text{SL}_2(\mathbb{Z})$. Most of the specific modular forms we consider in this paper are constructed from these Eisenstein series.

In [25], Schneider showed that if the value of the j-invariant $j(z)$ is algebraic, then z is either transcendental or a CM point. Using this result, Kanou [13] showed that for $k = 12$ or $k \geq 16$, at least one of the zeros of E_k in $\mathcal{F}$ is transcendental. Three years later, Kohnen [14] managed to show that, in fact, *all* the zeros of E_k in $\mathcal{F}$ (with the possible exceptions of ρ and i) are transcendental. To show this generalization, Kohnen utilized information on the location of zeros of Eisenstein series. In particular, he used a result from Rankin and Swinnerton-Dyer [20] stating that all the zeros of E_k in $\mathcal{F}$ lie on the lower arc $A = \left\{ z \in \mathcal{H} : |z| = 1, -\frac{1}{2} \leq x \leq 0 \right\}$. Similar results for other families of modular forms were later shown in [3, 4, 9, 11, 12].

In this paper, we first prove Theorem 1.1, showing how one can pass from information about locations of zeros to information about transcendence of zeros in certain scenarios. This theorem puts the arguments from all the previous papers [3, 4, 9, 11, 12, 14] in a more general context; although not explictly stated, the idea behind Theorem 1.1 essentially makes up the general philosophy behind these papers. In particular, Theorem 1.1 implies the main results of [3] and [4], as

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well as [11, Theorems 5, 6]. In the case of $p = 1$, Theorem 1.1 also recovers [11, Theorem 3], which in turn implies the main results of [9], [12], and [14].

In the following theorem, $\mathcal{F}(p)$ denotes the fundamental domain for $\Gamma_0(p)$, and $A(p)$ is a certain arc inside $\Gamma_0(p)$, both defined later in Subsection 4.1.

Theorem 1.1. *Fix $p \in \{1, 2, 3, 5, 7\}$. Let f be a nonzero modular form of weight k for $\Gamma_0(p)$ with rational Fourier coefficients at infinity. Suppose that all the zeros of f in $\mathcal{F}(p)$ lie on the arc $A(p)$. Then all the zeros of f are transcendental except for possibly the following:*

p	*Possible Algebraic Zeros*
1	ρ, i
2	$\frac{i\sqrt{2}}{2}$, $\frac{-1+i}{2}$, $\frac{\pm 1+i\sqrt{7}}{4}$
3	$\frac{i\sqrt{3}}{3}$, $\frac{-3+i\sqrt{3}}{6}$, $\frac{\pm 1+i\sqrt{11}}{6}$, $\frac{\pm 1+i\sqrt{2}}{3}$
5	$\frac{i\sqrt{5}}{5}$, $\frac{\pm 1+i\sqrt{19}}{10}$, $\frac{\pm 1+2i}{5}$, $\frac{\pm 2+i}{5}$, $\frac{\pm 3+i\sqrt{11}}{10}$, $\frac{-5+i\sqrt{5}}{10}$
7	$\frac{i\sqrt{7}}{7}$, $\frac{-7+i\sqrt{7}}{14}$, $\frac{\pm 6+i\sqrt{6}}{14}$, $\frac{\pm 5+i\sqrt{3}}{14}$, $\frac{\pm 4+2i\sqrt{3}}{14}$, $\frac{\pm 3+i\sqrt{19}}{14}$, $\frac{\pm 2+2i\sqrt{6}}{14}$, $\frac{\pm 1+3i\sqrt{3}}{14}$

This theorem can be applied to several different examples of modular forms. Specifically in Corollaries 4.1 and 4.2, we apply this theorem to obtain a complete list of algebraic zeros for $E_{k,p}^{\pm}(z) := \frac{1}{1 \pm p^{k/2}} \left(E_k(z) \pm p^{k/2} E_k(pz)\right)$. Later in Section 6, we will also apply this theorem to certain forms in the Miller basis.

Next, we show Theorem 1.2, which addresses modular forms of level one with zeros lying on the boundary of $\mathcal{F}$. We would like to point out that contrary to all the previous results in this area, this theorem shows transcendence for zeros lying on an unbounded curve.

Theorem 1.2. *Let f be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with rational Fourier coefficients at infinity. Suppose that all the zeros of f in $\mathcal{F}$ lie on the boundary of $\mathcal{F}$. Then aside from the finitely many* CM *points in $\mathscr{E}$, every zero of f in $\mathcal{F}$ is transcendental.*

Here, $\mathscr{E}$ denotes the set of CM *points with class group of exponent dividing* 2 *and discriminant odd or* -4. *Conditional on the non-existence of Siegel zeros, $\mathscr{E}$ is precisely the set of* 109 *CM points with discriminants given in the table from Lemma 2.2. Unconditionally, $\mathscr{E}$ could possibly also include the CM points arising from one additional fundamental discriminant.*

This theorem implies, for example, the transcendence of (all but finitely many of) the zeros of the cuspidal projection of the product of two Eisenstein series $\Delta_{k,\ell} := E_k E_\ell - E_{k+\ell}$. In fact, we conjecture in Conjecture 5.2 that, other than ρ and i, *all* the zeros of $\Delta_{k,\ell}$ are transcendental.

Finally, in Section 6, we address the Miller basis for S_k, the space of cusp forms of weight k and level one. We show transcendence of zeros for the first $2/9$ of the Miller basis, as well as for the last T forms in the Miller basis (for any given value of $T \geq 1$).

Theorem 1.3. *For $k \geq 4$ and $n(k) := \dim S_k$, let $\{g_{k,m}\}_{m=1}^{n(k)}$ denote the Miller basis for S_k.*
(1) For $1 \leq m < \frac{2n(k)-19}{9}$, other than ρ and i, the zeros of $g_{k,m}$ are all transcendental.
(2) Fix any $T \geq 1$. Then for sufficiently large k, other than i and ρ, the zeros of the last T forms in the Miller basis $g_{k,n(k)-T+1}, \ldots, g_{k,n(k)}$ are all transcendental.

In order to show Theorems 1.1, 1.2, and 1.3, we state a proposition for showing such transcendence results in general. Although not written down explicitly, the level one version of this proposition is implicit in [14], and we use the ideas in [10] to generalize it to modular forms of level N.

Proposition 1.4. *Given a nonzero modular form f over $\Gamma(N)$ with algebraic Fourier coefficients at infinity, all the zeros of f are either CM points or transcendental. Furthermore, suppose that the Fourier coefficients at infinity of f are rational and that $z_0 \in \mathrm{CM}_D$ is a zero of f. Then for every $z_1 \in \mathrm{CM}_D$, there exists some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $f(\gamma z_1) = 0$.*

We now give an overview of the paper. In Section 2, we discuss several preliminaries, including basic properties of the j-invariant, the j-invariant polynomial for modular forms, and the basics of CM theory. We also cite Lemma 2.2, which follows from our recent classification of class groups of exponent 2 [1]. Next, in Section 3, we prove Proposition 1.4. We then use this proposition to prove Theorem 1.1 in Section 4. We also apply this theorem to specific examples of modular forms in Corollaries 4.1 and 4.2. Next, we show Theorem 1.2 in Section 5 and apply it to show transcendence of zeros for cuspidal projections of products of Eisenstein series. We also state a conjecture about the zeros of these cuspidal projections of products of Eisenstein series. Finally, we show Theorem 1.3 in Section 6. We also state a conjecture about the zeros of the Miller basis.

2. Preliminaries

Let
$$\Delta := \frac{E_4^3 - E_6^2}{1728} \qquad j := \frac{E_4^3}{\Delta}$$
denote the modular discriminant and the Klein j-invariant, respectively.

For an even integer $k \geq 4$ write $k = 12\,n(k) + s_k$, where $s_k \in \{0, 4, 6, 8, 10, 14\}$ and $n(k) = \dim S_k$. Furthermore, set $a(k) \equiv k \mod 3$ where $a(k) \in \{0, 1, 2\}$, and $b(k) \equiv \frac{k}{2} \mod 2$ where $b(k) \in \{0, 1\}$. By the valence formula, a modular form of weight k must have zeros at the elliptic points ρ and i of certain orders. Here, $a(k)$ and $b(k)$ denote the orders at ρ and i respectively, as dictated by the valence formula. We call these zeros dictated by the valence formula the *trivial* zeros of a modular form of weight k.

We now state the following lemma, defining the j-invariant polynomials (see e.g. [24]).

Lemma 2.1. *Given a modular form f of level one and weight k, there is a unique polynomial P_f of degree exactly $n(k) - \mathrm{ord}_\infty(f)$ such that*
$$f = \Delta^{n(k)} E_4^{a(k)} E_6^{b(k)} P_f(j).$$

We call P_f the j-invariant polynomial of f.

Here, the non-trivial zeros z_0 of f correspond precisely to the zeros $j(z_0)$ of P_f. For this reason, it can often be a helpful perspective to view modular forms through the lens of their j-invariant polynomials. In most cases, it is easier to work with polynomials than with modular forms directly. Hence viewing modular forms in terms of their j-invariant polynomials can make it easier to address questions about the zeros of said modular forms (see e.g. [6, 12, 8, 24]).

Next, we review the basics of CM theory. The details here are all standard, and can be found in Cox [5]. We will use this theory to study the algebraic zeros of modular forms in Section 3.

A CM point z of discriminant D is the solution in $\mathcal{H}$ of a quadratic equation $az^2 + bz + c = 0$. Here $a, b, c \in \mathbb{Z}$ are chosen such that $a > 0$, $\gcd(a, b, c) = 1$, and $D = b^2 - 4ac < 0$. Let CM_D denote the set of CM points of discriminant D. For any discriminant D, it turns out that $\mathrm{SL}_2(\mathbb{Z})$ acts on CM_D via the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. Hence, we say that two CM_D points are equivalent if they are $\mathrm{SL}_2(\mathbb{Z})$ equivalent. Then there is a bijection between CM_D modulo $\mathrm{SL}_2(\mathbb{Z})$ and the class group $\mathrm{Cl}(\mathcal{O}_D)$, with $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{CM}_D$ inheriting the group structure of $\mathrm{Cl}(\mathcal{O}_D)$. Recall that $\mathcal{O}_D$ here is the imaginary quadratic order of discriminant D [5, Chapter 11, Section D]. Also, $h(D) = |\mathrm{Cl}(\mathcal{O}_D)|$ is known as the class number for discriminant D [18, A014600]. Finally, a discriminant D is *fundamental* if it is the discriminant of the quadratic field $\mathbb{Q}(\sqrt{D})$.

We also cite the following result concerning CM points lying on the boundary of $\mathcal{F}$, which we recently showed in [1]. This will be used later for Theorem 1.2.

Lemma 2.2 ([1, Theorem 1.2])**.** *Consider negative discriminants D. Then every CM_D point contained in $\mathcal{F}$ lies on the boundary of $\mathcal{F}$ if and only if $\mathrm{Cl}(\mathcal{O}_D)$ has exponent dividing 2 and D is odd or -4.*

Furthermore, conditional on the non-existence of Siegel zeros, such discriminants D are precisely those given in the following table. Unconditionally, this table could possibly also include the discriminants arising from one additional fundamental discriminant.

Class Group	*Discriminants*
$\{e\}$	$-3, -4, -7, -11, -19, -27, -43, -67, -163$
$\mathbb{Z}/2\mathbb{Z}$	$-15, -35, -51, -75, -91, -99, -115, -123, -147, -187, -235, -267, -403, -427$
$(\mathbb{Z}/2\mathbb{Z})^2$	$-195, -315, -435, -483, -555, -595, -627, -715, -795, -1435$
$(\mathbb{Z}/2\mathbb{Z})^3$	$-1155, -1995, -3003, -3315$

Finally, we cite two key lemmas concerning algebraicity of the j-invariant, which will be used throughout the paper. The first lemma follows from [5, Theorem 11.1, Proposition 13.2].

Lemma 2.3. *If $z \in \mathrm{CM}_D$ for some discriminant D, then $j(z)$ is an algebraic integer of degree $|\mathrm{Cl}(\mathcal{O}_D)|$. Moreover, if $w \in \mathrm{CM}_D$ as well, then $j(z)$ is Galois conjugate to $j(w)$ over $\mathbb{Q}(\sqrt{D})$.*

The second lemma from Schneider gives a way to show transcendence of certain $z \in \mathcal{H}$.

Lemma 2.4 ([25])**.** *If $z \in \mathcal{H}$, and $j(z)$ is algebraic, then either z is transcendental or a CM point.*

3. General Method

In this section, we outline the general method used to prove transcendence of zeros of modular forms, culminating in Proposition 1.4. This result combined with information about the locations of zeros will allow us to prove Theorems 1.1 and 1.2. We first prove the claim for level one modular forms.

Lemma 3.1. *Given a nonzero modular form f of level one with algebraic Fourier coefficients, all the zeros of f are either CM points or transcendental. Furthermore, if the Fourier coefficients are rational and some $z_0 \in \mathrm{CM}_D$ is a zero of f, every $z_1 \in \mathrm{CM}_D$ is also a zero of f.*

Proof. Fix a modular form f of weight k and level one with algebraic Fourier coefficients, and suppose that $f(z_0) = 0$.

If $z_0 = \rho$ or i, then z_0 is a CM point of discriminant -3 or -4, respectively. Furthermore, because the class numbers of discriminants -3 and -4 are both 1, $|\mathrm{Cl}(\mathcal{O}_D)| = 1$, and so $f(z_1) = 0$ for all $z_1 \in \mathrm{CM}_D$ (in this case, regardless of whether or not f has rational Fourier coefficients).

Otherwise, assume $z_0 \neq \rho, i$. Then the j-invariant polynomial $P_f(j)$ for f is non-constant and $P_f(j(z_0)) = 0$. Because f has algebraic Fourier coefficients, P_f also has algebraic coefficients, and so $j(z_0)$ is algebraic. Then by Lemma 2.4, z_0 is either a CM point or transcendental, finishing the proof of the first claim.

Now, suppose that the Fourier coefficients of f are rational and that $z_0 \in \mathrm{CM}_D$. Then by Lemma 2.3, for every $z_1 \in \mathrm{CM}_D$, $j(z_0)$ must be sent to $j(z_1)$ by some Galois automorphism σ fixing $\mathbb{Q}(\sqrt{D})$. Then using the fact that the coefficients of P_f are rational (since the Fourier coefficients of f are rational), applying σ to the equation $P_f(j(z_0)) = 0$ gives that likewise $P_f(j(z_1)) = 0$. This then yields $f(z_1) = 0$, as desired. □

We now prove Proposition 1.4.

Proposition 1.4. *Given a nonzero modular form f over $\Gamma(N)$ with algebraic Fourier coefficients at infinity, all the zeros of f are either CM points or transcendental. Furthermore, suppose that the Fourier coefficients at infinity of f are rational and that $z_0 \in \mathrm{CM}_D$ is a zero of f. Then for every $z_1 \in \mathrm{CM}_D$, there exists some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $f(\gamma z_1) = 0$.*

Proof. Let f be a modular form of weight k, level N, and with algebraic Fourier coefficients. Then the product over cosets of $\Gamma(N)$ in $\mathrm{SL}_2(\mathbb{Z})$,

$$F := \prod_{\gamma \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_k \gamma,$$

is a modular form for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ with algebraic Fourier coefficients [2, Theorem 4.1]. Note that any zero of f is also a zero of F. Thus by applying Lemma 3.1 to F, all the zeros of f are either CM points or transcendental.

Furthermore, if f has rational Fourier coefficients at infinity, then so does F. This is a consequence of [2, Theorem 3.3], which implies that for any fixed $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$F^\sigma = \prod_{\gamma \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} (f|_k\gamma)^\sigma = \prod_{\gamma \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f^\sigma|_k\gamma_\lambda,$$

where λ is a certain element of $(\mathbb{Z}/N\mathbb{Z})^\times$, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_\lambda \equiv \left(\begin{smallmatrix} a & \lambda b \\ \lambda^{-1}c & d \end{smallmatrix}\right) \pmod N$. Observe that the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_\lambda$ is a permutation on $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. Hence since $f^\sigma = f$,

$$F^\sigma = \prod_{\gamma \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f^\sigma|_k\gamma_\lambda = \prod_{\gamma' \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f|_k\gamma' = F,$$

which means that F has rational Fourier coefficients. So if any $z_0 \in \mathrm{CM}_D$ is a zero of f, then by Lemma 3.1, $F(z_1) = 0$ for all $z_1 \in \mathrm{CM}_D$. This means that for all $z_1 \in \mathrm{CM}_D$, $f(\gamma z_1) = 0$ for some $\gamma \in \Gamma(N)\backslash \mathrm{SL}_2(\mathbb{Z})$, as desired. $\square$

4. Modular forms with zeros on the arcs $A(p)$

We now prove Theorem 1.1, showing transcendence of zeros for modular forms with zeros on certain arcs. We also apply these results to several specific examples of modular forms.

4.1. **Proof of Theorem 1.1.**

For $p = 1, 2, 3, 5, 7$, let $\mathcal{F}(p)$ denote the following fundamental domains associated to $\Gamma_0(p)$.

$$\mathcal{F}(1) := \mathcal{F} = \left\{z \in \mathcal{H} : |z| \geq 1, -\frac{1}{2} \leq x \leq 0\right\} \cup \left\{z \in \mathcal{H} : |z| > 1, 0 < x < \frac{1}{2}\right\},$$

$$\mathcal{F}(2) := \left\{z \in \mathcal{H} : \left|z + \frac{1}{2}\right| \geq \frac{1}{2}, -\frac{1}{2} \leq x \leq 0\right\} \cup \left\{z \in \mathcal{H} : \left|z - \frac{1}{2}\right| > \frac{1}{2}, 0 < x < \frac{1}{2}\right\},$$

$$\mathcal{F}(3) := \left\{z \in \mathcal{H} : \left|z + \frac{1}{3}\right| \geq \frac{1}{3}, -\frac{1}{2} \leq x \leq 0\right\} \cup \left\{z \in \mathcal{H} : \left|z - \frac{1}{3}\right| > \frac{1}{3}, 0 < x < \frac{1}{2}\right\},$$

$$\mathcal{F}(5) := \left\{z \in \mathcal{H} : \left|z + \frac{1}{4}\right| \geq \frac{1}{4}, -\frac{1}{2} \leq x \leq 0\right\} \cup \left\{z \in \mathcal{H} : \left|z - \frac{1}{4}\right| > \frac{1}{4}, 0 < x < \frac{1}{2}\right\},$$

$$\mathcal{F}(7) := \left\{z \in \mathcal{H} : \left|z + \frac{1}{5}\right| \geq \frac{1}{5}, \left|z + \frac{3}{8}\right| \geq \frac{1}{8}, -\frac{1}{2} \leq x \leq 0\right\}$$
$$\cup \left\{z \in \mathcal{H} : \left|z - \frac{1}{5}\right| > \frac{1}{5}, \left|z - \frac{3}{8}\right| > \frac{1}{8}, 0 < x < \frac{1}{2}\right\}.$$

Let $A(p)$ denote the following arcs in each of these $\mathcal{F}(p)$.

$$A(1) := \left\{z \in \mathcal{H} : |z| = 1, -\frac{1}{2} \leq x \leq 0\right\},$$

$$A(2) := \left\{z \in \mathcal{H} : |z| = \frac{1}{\sqrt{2}}, -\frac{1}{2} \leq x < \frac{1}{2}\right\},$$

$$A(3) := \left\{z \in \mathcal{H} : |z| = \frac{1}{\sqrt{3}}, -\frac{1}{2} \leq x < \frac{1}{2}\right\},$$

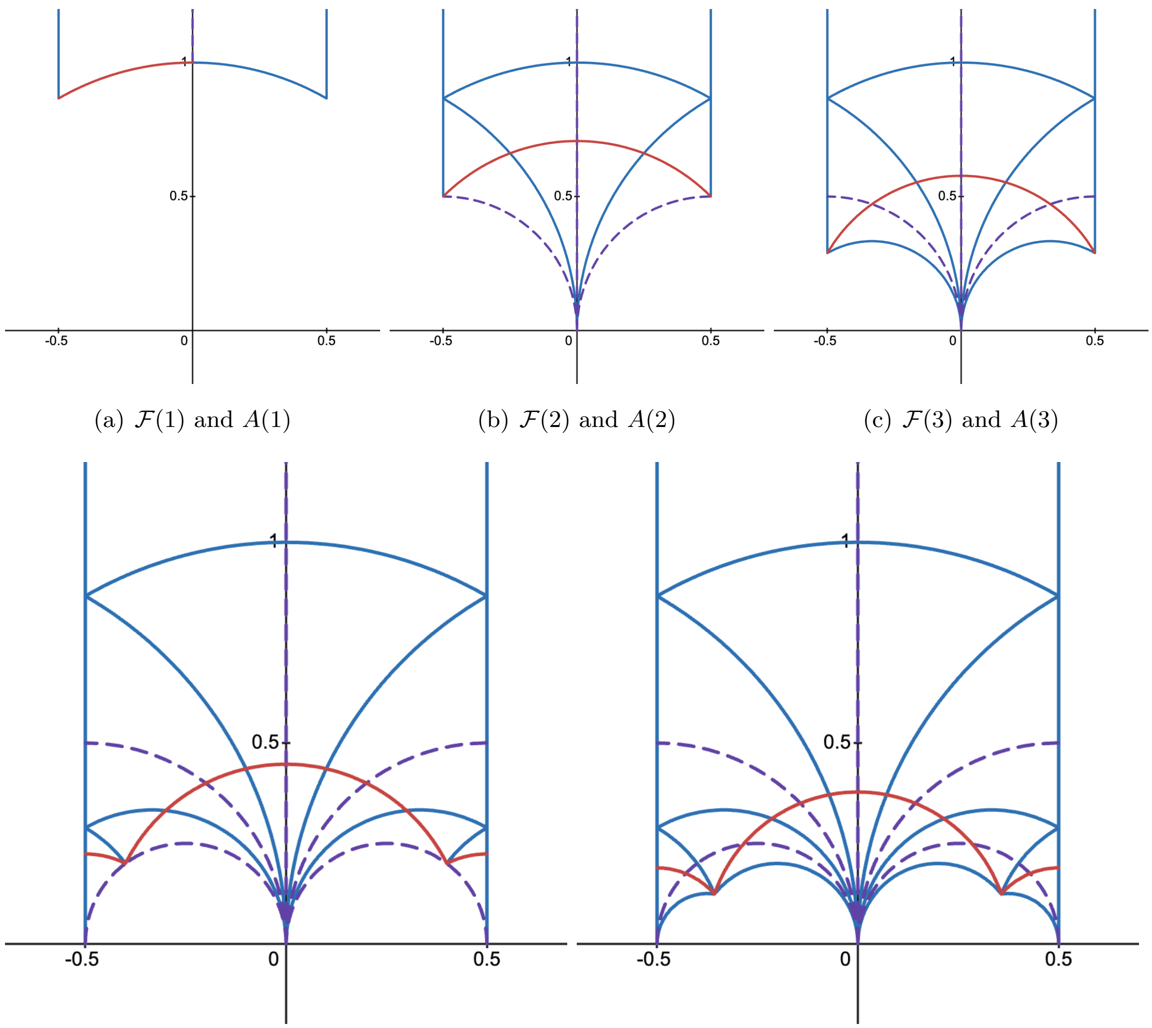
$$A(5) := \left\{z \in \mathcal{H} : |z| = \frac{1}{\sqrt{5}}, -\frac{2}{5} \leq x < \frac{2}{5}\right\} \cup \left\{z \in \mathcal{H} : \left|z + \frac{1}{2}\right| = \frac{1}{2\sqrt{5}}, -\frac{1}{2} \leq x < -\frac{2}{5}\right\}$$

$$\cup \left\{ z \in \mathcal{H} : \left| z - \frac{1}{2} \right| = \frac{1}{2\sqrt{5}}, \frac{2}{5} \leq x < \frac{1}{2} \right\},$$

$$A(7) := \left\{ z \in \mathcal{H} : |z| = \frac{1}{\sqrt{7}}, -\frac{5}{14} \leq x < \frac{5}{14} \right\} \cup \left\{ z \in \mathcal{H} : \left| z + \frac{1}{2} \right| = \frac{1}{2\sqrt{7}}, -\frac{1}{2} \leq x < -\frac{5}{14} \right\}$$

$$\cup \left\{ z \in \mathcal{H} : \left| z - \frac{1}{2} \right| = \frac{1}{2\sqrt{7}}, \frac{5}{14} \leq x < \frac{1}{2} \right\}.$$

We also provide a picture of these $\mathcal{F}(p)$ and $A(p)$ in Figure 4.1.



(a) $\mathcal{F}(1)$ and $A(1)$ (b) $\mathcal{F}(2)$ and $A(2)$ (c) $\mathcal{F}(3)$ and $A(3)$

(d) $\mathcal{F}(5)$ and $A(5)$ (e) $\mathcal{F}(7)$ and $A(7)$

FIGURE 4.1. The fundamental domains $\mathcal{F}(p)$ and corresponding arcs $A(p)$ for $p = 1, 2, 3, 5, 7$. The blue lines denote translates of the boundary of $\mathcal{F}(1)$, the purple dashed lines denote translates of the imaginary line in $\mathcal{F}(1)$, and the red arc denotes $A(p)$.

We now prove transcendence of zeros for modular forms over $\Gamma_0(p)$ with zeros lying on $A(p)$.

Theorem 1.1. *Fix $p \in \{1, 2, 3, 5, 7\}$. Let f be a nonzero modular form of weight k for $\Gamma_0(p)$ with rational Fourier coefficients at infinity. Suppose that all the zeros of f in $\mathcal{F}(p)$ lie on the arc $A(p)$. Then all the zeros of f are transcendental except for possibly the following:*

p	*Possible Algebraic Zeros*
1	ρ, i
2	$\frac{i\sqrt{2}}{2}$, $\frac{-1+i}{2}$, $\frac{\pm 1+i\sqrt{7}}{4}$
3	$\frac{i\sqrt{3}}{3}$, $\frac{-3+i\sqrt{3}}{6}$, $\frac{\pm 1+i\sqrt{11}}{6}$, $\frac{\pm 1+i\sqrt{2}}{3}$
5	$\frac{i\sqrt{5}}{5}$, $\frac{\pm 1+i\sqrt{19}}{10}$, $\frac{\pm 1+2i}{5}$, $\frac{\pm 2+i}{5}$, $\frac{\pm 3+i\sqrt{11}}{10}$, $\frac{-5+i\sqrt{5}}{10}$
7	$\frac{i\sqrt{7}}{7}$, $\frac{-7+i\sqrt{7}}{14}$, $\frac{\pm 6+i\sqrt{6}}{14}$, $\frac{\pm 5+i\sqrt{3}}{14}$, $\frac{\pm 4+2i\sqrt{3}}{14}$, $\frac{\pm 3+i\sqrt{19}}{14}$, $\frac{\pm 2+2i\sqrt{6}}{14}$, $\frac{\pm 1+3i\sqrt{3}}{14}$

Proof. The level $p = 1$ case of this theorem was shown in [11, Theorem 3]; we extend to levels $p > 1$. We give all the details for $p = 2$, and the arguments for $p = 3, 5, 7$ are nearly identical.

Let z_0 be a non-transcendental zero of f in $\mathcal{F}(2)$. Then we will show that z_0 is equal to $\frac{i\sqrt{2}}{2}$, $\frac{-1+i}{2}$, or $\frac{\pm 1+i\sqrt{7}}{4}$. By Proposition 1.4, z_0 must be a CM point, say of discriminant D. Moreover, *every* CM_D point must be $\mathrm{SL}_2(\mathbb{Z})$-equivalent to a zero of f. Now, by [5, Theorem 3.9], the identity element of CM_D in $\mathcal{F}(1)$ either lies on the left boundary L_1 of $\mathcal{F}(1)$ or the imaginary axis L_2 in $\mathcal{F}(1)$. Hence there exists a zero z_1 of f that lies on an $\mathrm{SL}_2(\mathbb{Z})$-translate of L_1 or L_2.

But there are only finitely many intersection points of $A(2)$ with the $\mathrm{SL}_2(\mathbb{Z})$ translates of L_1 and L_2 (see Figure 4.1). In particular, computing these intersection points tells us that

$$z_1 = \frac{i\sqrt{2}}{2},\ \frac{-1+i}{2},\ \text{or}\ \frac{\pm 1 + i\sqrt{7}}{4}.$$

The corresponding discriminants of these points are -8, -4, and -7 respectively. Each of these discriminants has class number 1, so in fact we must have $z_0 \equiv z_1 \mod \mathrm{SL}_2(\mathbb{Z})$. In particular, z_0 also lies on a $\mathrm{SL}_2(\mathbb{Z})$-translate of L_1 or L_2. Hence by the same argument as for z_1, we must have $z_0 = \frac{i\sqrt{2}}{2}$, $\frac{-1+i}{2}$, or $\frac{\pm 1+i\sqrt{7}}{4}$, completing the proof for $p = 2$.

For $p = 3$, the intersection points are $\frac{i\sqrt{3}}{3}$, $\frac{-3+i\sqrt{3}}{6}$, $\frac{\pm 1+i\sqrt{11}}{6}$, and $\frac{\pm 1+i\sqrt{2}}{3}$. The corresponding discriminants of these points are -12, -3, -11, and -8. And each of these discriminants has class number 1. Hence by the same argument as for $p = 2$, these are the only possible non-transcendental zeros of f.

For $p = 5$, the intersection points are $\frac{i\sqrt{5}}{5}$, $\frac{\pm 1+i\sqrt{19}}{10}$, $\frac{\pm 1+2i}{5}$, $\frac{\pm 2+i}{5}$, $\frac{\pm 3+i\sqrt{11}}{10}$, and $\frac{-5+i\sqrt{5}}{10}$. The corresponding discriminants are -20, -19, -16, -4, -11, and -20. Other than -20, these discriminants all have class number 1. Discriminant -20 has class number 2, so in this case, z_0 is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to either $\frac{i\sqrt{5}}{5}$ or $\frac{-5+i\sqrt{5}}{10}$ (which are both already contained in the list of intersection points).

For $p = 7$, the intersection points are $\frac{i\sqrt{7}}{7}$, $\frac{\pm 7+i\sqrt{7}}{14}$, $\frac{\pm 6+i\sqrt{6}}{14}$, $\frac{\pm 5+i\sqrt{3}}{14}$, $\frac{\pm 4+2i\sqrt{3}}{14}$, $\frac{\pm 3+i\sqrt{19}}{14}$, $\frac{\pm 2+2i\sqrt{6}}{14}$, and $\frac{\pm 1+3i\sqrt{3}}{14}$. The corresponding discriminants are -28, -7, -24, -3, -12, -19, -24, and -27.

Other than -24, these all have class number 1. Discriminant -24 has class number 2, so in this case, z_0 is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to either $\frac{\pm 6+i\sqrt{6}}{14}$ or $\frac{\pm 2+2i\sqrt{6}}{14}$ (which are both already contained in the list of intersection points). $\square$

4.2. **Applications of Theorem 1.1.**

We note that Theorem 1.1 recovers the main results of [4] and [3], as well as [11, Theorems 5, 6]. For the case of $p = 1$, Theorem 1.1 recovers [11, Theorem 3], which in turn implies the main results of [9], [12], and [14].

We give several additional interesting consequences of Theorem 1.1 for modular forms of higher level. In particular, let

$$E_{k,2}^{-}(z) := \frac{1}{1-2^{k/2}}\left(E_k(z) - 2^{k/2}E_k(2z)\right),$$

which is the Eisenstein series associated with $\Gamma_0(2)$. In this case, it was shown by Oh [19] that all the zeros of $E_{k,2}^{-}$ lie on $A(2)$. As a part of the proof, Oh showed that $E_{k,2}^{-}\left(\frac{i\sqrt{2}}{2}\right) = 0$ if and only if $k \equiv 0 \pmod 4$ and $E_{k,2}^{-}\left(\frac{-1+i}{2}\right) = 0$ if and only if $k \equiv 0, 2, 6 \pmod 8$. Using Theorem 1.1, we show that in fact, these are the only two possible algebraic zeros of $E_{k,2}^{-}$.

Corollary 4.1. *Other than $\frac{i\sqrt{2}}{2}$ and $\frac{-1+i}{2}$, every zero of $E_{k,2}^{-}$ in $\mathcal{F}(2)$ is transcendental.*

Proof. From Theorem 1.1, we know that $\frac{i\sqrt{2}}{2}$, $\frac{-1+i}{2}$, and $\frac{\pm 1+i\sqrt{7}}{4}$ are the only possible algebraic zeros of $E_{k,2}^{-}$. We show that in fact $\alpha := \frac{\pm 1+i\sqrt{7}}{4}$ cannot be a zero using the structure of $E_{k,2}^{-}$. First, note that there exists $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(2\alpha) = \alpha$; in particular

$$\begin{pmatrix} 0 & -1 \\ 1 & \mp 1 \end{pmatrix}\left(\frac{\pm 1 + i\sqrt{7}}{2}\right) = \frac{\pm 1 + i\sqrt{7}}{4}.$$

This means that

$$\begin{aligned} E_{k,2}^{-}(\alpha) &= \frac{1}{1-2^{k/2}}\left(E_k(\alpha) - 2^{k/2}E_k(2\alpha)\right) \\ &= \frac{1}{1-2^{k/2}}\left(E_k(\gamma(2\alpha)) - 2^{k/2}E_k(2\alpha)\right) \\ &= \frac{1}{1-2^{k/2}}\left((2c\alpha+d)^k E_k(2\alpha) - 2^{k/2}E_k(2\alpha)\right) \\ &= \frac{E_k(2\alpha)}{1-2^{k/2}}\left((2c\alpha+d)^k - 2^{k/2}\right). \end{aligned}$$

The above expression vanishes if and only if $E_k(2\alpha) = 0$ or $(2c\alpha + d)^k = 2^{k/2}$.

For the first case, notice that α is not an $\mathrm{SL}_2(\mathbb{Z})$-translate of ρ or i, so it cannot be a zero of E_k by [14]. The second case requires that $(2c\alpha+d)^{k/2} = \left(\frac{\mp 1+i\sqrt{7}}{2}\right)^k = 2^{k/2}$. But this would mean that $\left(\frac{\mp 1+i\sqrt{7}}{2}\right)^k$ is rational, which is impossible by [17, Theorem 3.7]. Thus in all cases, $\alpha = \frac{\pm 1+i\sqrt{7}}{4}$ is not a zero of $E_{k,2}^{-}$. $\square$

For p prime, the Eisenstein series

$$E_{k,p}^{+}(z) := \frac{1}{1+p^{k/2}}\left(E_k(z) + p^{k/2}E_k(pz)\right)$$

is a modular form of weight k on the Fricke group $\Gamma_0^+(p) := \Gamma_0(p) \cup \Gamma_0(p)W_p$, where

$$W_p := \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right).$$

Denote $\mathcal{F}^+(p)$ as the standard fundamental domain of $\Gamma_0^+(p)$, (the part of $\mathcal{F}(p)$ lying above $A(p)$; see [16, 26, 15]). For $p = 2, 3, 5, 7$, the zeros of $E_{k,p}^+$ in $\mathcal{F}^+(p)$ were shown to lie on the left half of $A(p)$ in [16, 26, 15].

For $p = 2, 3$, Gun and Saha have already shown the transcendence of all but finitely many of the zeros of $E_{k,2}^+$ and $E_{k,3}^+$ in [11, Theorems 5, 6]. We improve on this result, further eliminating possible algebraic zeros and showing similar statements for $p = 5, 7$.

Corollary 4.2. *Fix $p \in \{1, 2, 3, 5, 7\}$. Then all zeros of $E_{k,p}^+$ in $\mathcal{F}^+(p)$ are transcendental except possibly for the following elliptic points:*

p	*Possible Algebraic Zeros*
1	$\rho,\ i$
2	$\frac{i\sqrt{2}}{2},\ \frac{-1+i}{2}$
3	$\frac{i\sqrt{3}}{3},\ \frac{-3+i\sqrt{3}}{6}$
5	$\frac{i\sqrt{5}}{5},\ \frac{-2+i}{5},\ \frac{-5+i\sqrt{5}}{10}$
7	$\frac{i\sqrt{7}}{7},\ \frac{-5+i\sqrt{3}}{14},\ \frac{-7+i\sqrt{7}}{14}$

Proof. Observe that $\Gamma_0(p) \subseteq \Gamma_0^+(p)$. Furthermore, due to the symmetry of $\Gamma_0^+(p)$, all the zeros of $E_{k,p}^+$ in $\mathcal{F}(p)$ lie on the arc $A(p)$. Therefore we can apply Theorem 1.1. To avoid redundancy, we will consider only the zeros in $\mathcal{F}^+(p)$, i.e. the ones lying on the left half of $A(p)$. Now, we consider each value of p separately.

In the following, we use the same proof strategy and notation as in Corollary 4.1.

Case $p = 1$: Here, $E_{k,1}^+$ is simply the Eisenstein series over the full modular group $\mathrm{SL}_2(\mathbb{Z})$, and so this case was already shown in [14].

Case $p = 2$: The proof is essentially identical to Corollary 4.1.

Case $p = 3$: From Theorem 1.1, the possible CM zeros are $\frac{i\sqrt{3}}{3}$, $\frac{-3+i\sqrt{3}}{6}$, $\frac{-1+i\sqrt{11}}{6}$, and $\frac{-1+i\sqrt{2}}{3}$. Notice that for $\alpha = \frac{-1+i\sqrt{11}}{6}, \frac{-1+i\sqrt{2}}{3}$,

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\left(\frac{-1+i\sqrt{11}}{2}\right) = \frac{-1+i\sqrt{11}}{6}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\left(-1+i\sqrt{2}\right) = \frac{-1+i\sqrt{2}}{3}.$$

Hence by a similar argument as in Corollary 4.1 we have

$$E_{k,3}^{+}(\alpha) = \frac{E_k(3\alpha)}{1+3^{k/2}}\left((3c\alpha+d)^k + 3^{k/2}\right).$$

Thus α can only be a zero of $E_{k,3}^{+}$ if $(3c\alpha+d)^k = -3^{k/2}$ (as α is not a zero of E_k). But the left hand side of this equation is either $\left(\frac{1+i\sqrt{11}}{2}\right)^k$ or $(1+i\sqrt{2})^k$, which are never rational by [17, Theorem 3.7]. Hence α is not a zero of $E_{k,3}^{+}$.

Case $p=5$: From Theorem 1.1, the possible CM zeros are $\frac{i\sqrt{5}}{5}$, $\frac{-1+i\sqrt{19}}{10}$, $\frac{-1+2i}{5}$, $\frac{-2+i}{5}$, $\frac{-3+i\sqrt{11}}{10}$, and $\frac{-5+i\sqrt{5}}{10}$. Notice that for $\alpha = \frac{-1+i\sqrt{19}}{10}, \frac{-1+2i}{5}, \frac{-3+i\sqrt{11}}{10}$,

$$\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\left(\frac{-1+i\sqrt{19}}{2}\right) = \frac{-1+i\sqrt{19}}{10}, \quad \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}(-1+2i) = \frac{-1+2i}{5},$$
$$\begin{pmatrix}0 & -1\\ 1 & 3\end{pmatrix}\left(\frac{-3+i\sqrt{11}}{2}\right) = \frac{-3+i\sqrt{11}}{10}.$$

Here, α can only be a zero of $E_{k,5}^{+}$ if $(5c\alpha+d)^k = -5^{k/2}$. The left hand side of the equation is either $\left(\frac{1+i\sqrt{19}}{2}\right)^k$, $(1+2i)^k$, or $\left(\frac{3+i\sqrt{11}}{2}\right)^k$, which are never rational by [17, Theorem 3.7]. Hence α is not a zero of $E_{k,5}^{+}$.

Case $p=7$: From Theorem 1.1, the possible CM zeros are $\frac{i\sqrt{7}}{7}$, $\frac{-1+i\sqrt{7}}{14}$, $\frac{-6+i\sqrt{6}}{14}$, $\frac{-5+i\sqrt{3}}{14}$, $\frac{-4+2i\sqrt{3}}{14}$, $\frac{-3+i\sqrt{19}}{14}$, $\frac{-2+2i\sqrt{6}}{14}$, and $\frac{-1+3i\sqrt{3}}{14}$. Notice that for $\alpha = \frac{-1+3i\sqrt{3}}{14}, \frac{-1+i\sqrt{6}}{7}, \frac{-3+i\sqrt{19}}{14}, \frac{-2+i\sqrt{3}}{7}, \frac{-6+i\sqrt{6}}{14}$,

$$\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\left(\frac{-1+3i\sqrt{3}}{2}\right) = \frac{-1+3i\sqrt{3}}{14}, \quad \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}\left(-1+i\sqrt{6}\right) = \frac{-1+i\sqrt{6}}{7},$$
$$\begin{pmatrix}0 & -1\\ 1 & 3\end{pmatrix}\left(\frac{-3+i\sqrt{19}}{2}\right) = \frac{-3+i\sqrt{19}}{14}, \quad \begin{pmatrix}0 & -1\\ 1 & 4\end{pmatrix}\left(-2+i\sqrt{3}\right) = \frac{-2+i\sqrt{3}}{7},$$
$$\begin{pmatrix}-1 & -3\\ 2 & 5\end{pmatrix}\left(\frac{-6+i\sqrt{6}}{2}\right) = \frac{-6+i\sqrt{6}}{14}.$$

Here, α can only be a zero of $E_{k,7}^{+}$ if $(7c\alpha+d)^k = -7^{k/2}$. The left hand side of this equation is one of $\left(\frac{1+3i\sqrt{3}}{2}\right)^k$, $\left(1+i\sqrt{6}\right)^k$, $\left(\frac{3+i\sqrt{19}}{2}\right)^k$, $(2+3i)^k$, or $(-1+i\sqrt{6})^k$, which are never rational by [17, Theorem 3.7]. Hence α is not a zero of $E_{k,7}^{+}$. □

5. Modular forms with zeros on the boundary of $\mathcal{F}$

We next consider level one modular forms with zeros all lying on the boundary of $\mathcal{F}$. The following theorem follows immediately from Proposition 1.4 (with level $N=1$) and Lemma 2.2 (with its corresponding table).

Theorem 1.2. *Let f be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with rational Fourier coefficients at infinity. Suppose that all the zeros of f in $\mathcal{F}$ lie on the boundary of $\mathcal{F}$. Then aside from the finitely many* CM *points in $\mathscr{E}$, every zero of f in $\mathcal{F}$ is transcendental.*

Here, $\mathscr{E}$ denotes the set of CM *points with class group of exponent dividing* 2 *and discriminant odd or* -4*. Conditional on the non-existence of Siegel zeros,* $\mathscr{E}$ *is precisely the set of* 109 *CM points with discriminants given in the table from Lemma 2.2. Unconditionally,* $\mathscr{E}$ *could possibly also include the CM points arising from one additional fundamental discriminant.*

One example of modular forms to which we can apply this theorem is the cuspidal projection

$$\Delta_{k,\ell} := E_k E_\ell - E_{k+\ell}$$

of weight $k+\ell$. In [22] and [27], Reitzes-Vulakh-Young and Xue-Zhu showed that for $k, \ell \geq 4$ with $k+\ell \neq 8, 10, 14$, all the zeros of $\Delta_{k,\ell}$ lie on the boundary of $\mathcal{F}$. (Note that this covers all possible cases; when $k+\ell = 8, 10, 14$, $\Delta_{k,\ell}$ is identically 0.) Hence we can apply Theorem 1.2 in particular to these cuspidal projections $\Delta_{k,\ell}$.

Corollary 5.1. *Let $k, \ell \geq 4$ be even with $k+\ell \neq 8, 10, 14$. Then aside from the finitely many CM points in $\mathscr{E}$ (see Theorem 1.2), every zero of $\Delta_{k,\ell}$ in $\mathcal{F}$ is transcendental.*

We speculate that, other than ρ and i, *none* of the zeros of $\Delta_{k,\ell}$ are transcendental. We formulate this in the following conjecture.

Conjecture 5.2. *Fix $k, \ell \geq 4$ such that $k+\ell \neq 8, 10, 14$. Then aside from ρ and i, every zero of $\Delta_{k,\ell}$ in $\mathcal{F}$ is transcendental.*

Assuming non-existence of Siegel zeros, it turns out that Conjecture 5.2 would also follow from the irreducibility of the j-invariant polynomials for $\Delta_{k,\ell}$.

Conjecture 5.3. *For any $k, \ell \geq 4$, the j-invariant polynomial of $\Delta_{k,\ell}$ is either constant or irreducible over $\mathbb{Q}$.*

We have verified Conjectures 5.2 and 5.3 computationally for all $k+\ell \leq 1000$ [23]. For $z_0 \in \mathrm{CM}_D$, the minimal polynomial of $j(z_0)$ is precisely the Hilbert class polynomial (sometimes known as the ring class polynomial) of discriminant D [5, Chapter 11, Section D]. Hence to find the algebraic zeros of f, one just needs compute the Hilbert class polynomials dividing the j invariant polynomial for f. This strategy is what made all of our computations possible.

Finally, we note why Conjecture 5.3 implies Conjecture 5.2 (assuming the non-existence of Siegel zeros). Under these assumptions, it turns that any counterexample to Conjecture 5.2 would necessarily have weight $k+\ell \leq 122$. And we have already verified Conjecture 5.2 for these finitely many cases. Suppose that $z_0 \in \mathscr{E} \backslash \{\rho, i\}$, say with discriminant D, is such that $\Delta_{k,\ell}(z_0) = 0$. By the above paragraph, this means that the Hilbert class polynomial for D, HCP_D, divides the j-invariant polynomial for $\Delta_{k,\ell}$, $P_{\Delta_{k,\ell}}$. Assuming Conjecture 5.3, this would then mean that, in particular, $\deg \mathrm{HCP}_D = \deg P_{\Delta_{k,\ell}}$. However, we have the upper bound $\deg \mathrm{HCP}_D = |\mathrm{Cl}(\mathcal{O}_D)| \leq 8$ by Lemma 2.2, and the lower bound $\deg P_{\Delta_{k,\ell}} = n(k+\ell) - 1 \geq \frac{k+\ell-14}{12} - 1$ by Lemma 2.1. Comparing these two bounds, we see that $\deg \mathrm{HCP}_D = \deg P_{\Delta_{k,\ell}}$ can only occur for $k+\ell \leq 122$, as claimed.

Note that in our application of Lemma 2.1, we used the fact that $\mathrm{ord}_\infty(\Delta_{k,\ell}) = 1$. This follows immediately from the fact that the q-coefficient of $\Delta_{k,\ell}$, $2\left(\frac{k+\ell}{B_{k+\ell}} - \frac{k}{B_k} - \frac{\ell}{B_\ell}\right)$, never vanishes for $k+\ell \neq 8, 10, 14$ [7, paragraph following Conjecture 7.1].

6. Modular forms in the Miller basis

Recall that for $1 \leq m \leq n(k)$, the weight k, level one modular form

$$g_{k,m}(z) = q^m + O(q^{n(k)+1})$$

is uniquely determined. Furthermore, $\{g_{k,m}\}_{m=1}^{n(k)}$ forms a basis for S_k, called the Miller basis. In this last section, we show transcendence of zeros for the first 2/9 of the Miller basis, as well as the last T forms in the Miller basis (for any given value of $T \geq 1$).

Theorem 1.3. *For $k \geq 4$ and $n(k) := \dim S_k$, let $\{g_{k,m}\}_{m=1}^{n(k)}$ denote the Miller basis for S_k.*
(1) For $1 \leq m < \frac{2n(k)-19}{9}$, other than ρ and i, the zeros of $g_{k,m}$ are all transcendental.
(2) Fix any $T \geq 1$. Then for sufficiently large k, other than ρ and i, the zeros of the last T forms in the Miller basis $g_{k,n(k)-T+1}, \ldots, g_{k,n(k)}$ are all transcendental.

Proof. **(1).** By [21, Theorem 4.1], all the zeros of these $g_{k,m}$ lie on the arc $A(1)$. Thus by Theorem 1.1, all the zeros other than ρ and i are transcendental.

(2). Fix $T \geq 1$. Then for $k \geq 4$, let $f_k := g_{k,n(k)-T+1}$ denote the Tth-to-last form in the Miller basis for S_k. It suffices to show the desired result for this f_k (then one can choose a sufficiently large k such that the result holds for the finitely many $1 \leq T' \leq T$).

By [24, Theorem 1.1], the T non-trivial zeros $\{z_{k,j}\}_{1\leq j\leq T}$ of f_k in $\mathcal{F}$ satisfy

$$z_{k,j} = \frac{i}{2\pi}\log(2kw_{T,j}) + O\left(\frac{1}{k}\right)$$

where the $\{w_{T,j}\}_{1\leq j\leq T}$ denote the inverses of the roots of $\sum_{n=0}^{T}\frac{t^n}{n!}$. In particular, this means that $\mathrm{Im}(z_j) \to \infty$ as $k \to \infty$, uniformly over $1 \leq j \leq T$.

On the other hand, observe that there are only a finite number of possible non-transcendental zeros for the f_k. Suppose that a non-trivial zero z_j in $\mathcal{F}$ is not transcendental. Then, as f has rational Fourier coefficients, by Proposition 1.4, $z_j \in \mathrm{CM}_D$ for some D and furthermore all points in CM_D are zeros of f. As f has exactly T non-trivial zeros in $\mathcal{F}$, this means that we must have $|\mathrm{Cl}(\mathcal{O}_D)| \leq T$. However, there are only finitely many such discriminants D, and so only finitely many possible non-transcendental zeros for the f_k. In particular, the imaginary part of these possible non-transcendental zeros is bounded.

Hence for sufficiently large k, every zero of f_k will be transcendental. □

We note here that part (2) of this theorem in fact holds for any rational linear combination of the last T forms in the Miller basis; see [24, Theorem 1.1]. We state the theorem as above for simplicity.

Inspired by Theorem 1.3, we finish with the following conjecture. We have verified this conjecture for all weights $k \leq 3000$ [23].

Conjecture 6.1. *Let f be any form in the Miller basis for S_k. Then other than ρ and i, every zero of f is transcendental.*

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