ARE SETS WITH A GIVEN MULTIPLICATIVE STRUCTURE GUARANTEED TO HAVE A DENSITY?

ERICK ROSS

Abstract. In this paper, we study the question of whether or not sets of natural numbers with a given multiplicative structure are guaranteed to have a density. We consider several different types of multiplicative structure (multiplicatively closed sets, sets of multiples, saturated sets, and multiplicatively closed saturated sets), and several different different types of density (natural density, logarithmic density, and multiplicative density). Many of these cases have been studied before; in this paper, we finish off the problem, answering the above question in every case.

1. Introduction

In this paper, we study the titular question: Are sets of natural numbers with a given multiplicative structure are guaranteed to have a density? We answer this question for four different types of multiplicative structure (multiplicatively closed sets, sets of multiples, saturated sets, and multiplicatively closed saturated sets), and three different types of density (natural density, logarithmic density, and multiplicative density).

The investigation of this question started in 1934 when Chowla conjectured that any set of multiples would have a natural density [3]. This conjecture, however, turns out to not be true; Besicovitch provided a counterexample the next year [2]. Nevertheless, many interesting problems remained concerning the density of sets of multiples. These problems have turned out to be of great interest, with a rich history of research over the years (e.g. see [8, 12, 14] for the state-of-the-art at the end of the 20'th century). Erdős, especially, took interest in these problems (e.g. see [4, 5, 6, 7, 8, 9, 10]), and along with Davenport proved the most well-known result in the area: the Davenport-Erdős theorem states that sets of multiples are guaranteed to have an logarithmic density [4].

As one can see from the summary above, most of the research so far in this area has been focused on sets of multiples. In this paper, our goal is to extend this study to other multiplicative structures. In particular, we show the following theorem.

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Theorem 1.1. *The following table answers the titular question: Are sets of natural numbers with a given multiplicative structure guaranteed to have a density?*

	Natural density	*Logarithmic density*	*Multiplicative density*
Multiplicatively closed sets	*No*	*No*	*No*
Sets of multiples	*No*	*Yes*	*Yes*
Saturated sets	*No*	*Yes*	*Yes*
Multiplicatively closed saturated sets	*Yes*	*Yes*	*Yes*

We have stated the entire table here for context. However, note that the main novelty of this paper comes from the problems regarding multiplicatively closed sets. The problems from most of the rest of the table have either already been studied in previous works, or quickly follow from known results.

The results of Theorem 1.1 for multiplicatively closed sets are quite surprising, at least to the author. We were expecting the opposite result, partially because of the close analogy with the Davenport-Erdős theorem. In the May 2025 preprint of [13], we had even formally stated (the incorrect) Conjecture 7.3: that every multiplicatively closed set $B \subseteq \mathbb{N}$ has a logarithmic density.

2. Basic Definitions

Recall that a set $B \subseteq \mathbb{N}$ is *multiplicatively closed* if $a, b \in B \implies ab \in B$. Similarly, $B \subseteq \mathbb{N}$ is called a *set of multiples* if $b \in B, n \in \mathbb{N} \implies nb \in B$. Lastly, a set $B \subseteq \mathbb{N}$ is *saturated* if $b \in B \implies a \in B$ for all $a \mid b$. Note that the four multiplicative structures we are considering in this paper make up all the nontrivial combinations of these definitions (since saturated sets of multiples are either $\varnothing$ or $\mathbb{N}$, and sets of multiples are already multiplicatively closed). Also note that saturated sets are precisely the complements of sets of multiples.

Next, recall that the *natural density* of a set $B \subseteq \mathbb{N}$ is defined as

$$d(B) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} 1_B(n),$$

where 1_B denotes the indicator function $1_B(n) := \mathbb{1}_{n\in B}$. Similarly, we define the *logarithmic density* of a set $B \subseteq \mathbb{N}$ to be

$$\delta(B) := \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n} 1_B(n).$$

Lastly, we define the *multiplicative density* of a set $B \subseteq \mathbb{N}$ to be

$$\Delta(B) := \lim_{y\to\infty} \left(\sum_{n\in\mathbb{N}_y} \frac{1}{n} \right)^{-1} \sum_{n\in\mathbb{N}_y} \frac{1}{n} 1_B(n),$$

where $\mathbb{N}_y$ denotes the set of all y-smooth numbers (i.e. the natural numbers n such that $p \le y$ for all primes $p \mid n$.)

Of course, the above limits are not guaranteed to exist in general. So we also define the upper and lower densities in each case to be the corresponding lim sup and lim inf, respectively. For example, the upper and lower natural densities are defined as

$$\overline{d}(B) := \limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} 1_B(n) \qquad \text{and} \qquad \underline{d}(B) := \liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} 1_B(n),$$

and the natural density $d(B)$ exists if and only if $\overline{d}(B) = \underline{d}(B)$.

We also take note here of the well-known inequality $\underline{d}(B) \le \underline{\delta}(B) \le \overline{\delta}(B) \le \overline{d}(B)$ [1, Corollary 1.12]. In particular, this inequality means that if the set B has a natural density, then it necessarily also has a logarithmic density (with the same value).

3. Sets of multiples

In 1935, Besicovitch managed to construct a set of multiples without a natural density [2]. For the convenience of the reader, we repeat Besicovitch's construction here (modified slightly for simplicity).

Proposition 3.1. *There exists a set of multiples without a natural density.*

Proof. For any set $B \subseteq \mathbb{N}$, let $d(B,x) := \frac{1}{x}\#\{n \le x : n \in B\}$. (So that the upper and lower natural densities of B are $\overline{d}(B) := \limsup_{x\to\infty} d(B,x)$ and $\underline{d}(B) := \liminf_{x\to\infty} d(B,x)$, respectively.)

Then for $k \ge 1$, let R_k denote the range of integers $[2^k, 2^{k+1})$, and let $\mathbb{N}R_k$ denote the set of all multiples of elements of R_k. Besicovitch showed in [2, Theorem 1] that the $d(\mathbb{N}R_k)$ all exist, and that $\liminf d(\mathbb{N}R_k) = 0$. Then observe that $1_{\mathbb{N}R_k}(n)$ is $2^{k+1}!$-periodic (i.e. $n \in \mathbb{N}R_k$ if and only if $n + 2^{k+1}! \in \mathbb{N}R_k$). Hence setting $d_k := d(\mathbb{N}R_k, 2^{k+1}!)$, we have that $d(\mathbb{N}R_k, x) = d_k$ for each x a multiple of $2^{k+1}!$. Note this also implies that $d_k = d(\mathbb{N}R_k)$.

Now, choose indices $k_1 < k_2 < k_3 < \dots$ large enough such that

$$d_{k_1} \le \frac{1}{8}$$

$$d_{k_2} \le \frac{1}{16} \quad \text{and} \quad 2^{k_2} > 2^{k_1+1}!$$

$$d_{k_3} \leq \frac{1}{32} \quad \text{and} \quad 2^{k_3} > 2^{k_2+1}!$$

$$\vdots$$

Then let B be the set of multiples $B := \mathbb{N}(R_{k_1} \cup R_{k_2} \cup \ldots)$, and we will show that B does not have a density.

For $i \geq 1$, consider $x_i = 2^{k_i+1}$. Since $R_{k_i} = [2^{k_i}, 2^{k_i+1}) \subseteq B$, we have that $d(B, x_i) \geq \frac{1}{2}$ for all i and so $\overline{d}(B) \geq \frac{1}{2}$. Alternatively, for $i \geq 1$, consider $x_i = 2^{k_i+1}!$. Note that

$$\begin{aligned} B &= \mathbb{N}R_{k_1} \cup \mathbb{N}R_{k_2} \cup \ldots, \\ \text{so} \quad d(B, x_i) &\leq d(\mathbb{N}R_{k_1}, x_i) + d(\mathbb{N}R_{k_2}, x_i) + \ldots \end{aligned}$$

Now, for all $j \geq i+1$, we have $2^{k_j} > x_i$, so $d(\mathbb{N}R_{k_j}, x_i) = 0$ And for all $j \leq i$, we have that x_i is a multiple of $2^{k_j+1}!$, so $d(\mathbb{N}R_{k_j}, x_i) = d_{k_j}$. This means that

$$\begin{aligned} d(B, x_i) &\leq d(\mathbb{N}R_{k_1}, x_i) + d(\mathbb{N}R_{k_2}, x_i) + \ldots \\ &= d_{k_1} + \ldots + d_{k_i} \\ &\leq \frac{1}{8} + \ldots + \frac{1}{2^{i+2}} \\ &\leq \frac{1}{4} \end{aligned}$$

for all i and so $\underline{d}(B) \leq \frac{1}{4}$.

Thus $\underline{d}(B) < \overline{d}(B)$, and so B does not have a density. $\square$

In 1937 [4], Davenport and Erdős proved what is now known as the Davenport-Erdős theorem: that every set of multiples has a logarithmic density. We give a brief sketch of the argument they used.

One can write any set of multiples B as $B = \mathbb{N}A$ for some set of generators A. Then write $A = \{a_1, a_2, \ldots\}$ (in increasing order) and define

$$\begin{aligned} A_1 &= \frac{1}{a_1} \\ A_2 &= \frac{1}{a_2} - \frac{1}{[a_1, a_2]} \\ &\vdots \\ A_k &= \frac{1}{a_k} - \sum_{j<k} \frac{1}{[a_j, a_k]} + \sum_{i<j<k} \frac{1}{[a_i, a_j, a_k]} - \ldots, \end{aligned}$$

where $[c_1, c_2, \ldots, c_n] := \mathrm{lcm}(c_1, c_2, \ldots, c_n)$.

Using an inclusion-exclusion argument, one can see that A_k represents the density of the natural numbers divisible by a_k, but not by any of $a_1, \dots a_{k-1}$. Hence it is reasonable to guess that the (logarithmic) density of $B = \mathbb{N}A$ should be $A_0 := \sum_{k\geq 1} A_k$.

To prove this guess, one would need to show that the growth rate of $\sum_{n\leq x} \frac{1}{n} 1_B(n)$ is $A_0 \log x + o(\log x)$. Davenport and Erdős were able to compute this growth rate using the Dirichlet series L-function for 1_B:

$$L(1_B, s) := \sum_{n\geq 1} \frac{1_B(n)}{n^s}.$$

In particular, by a Tauberian theorem (Wiener-Ikehara), the growth rate of $\sum_{n\leq x} \frac{1_B(n)}{n}$ can be determined by calculating the residue at the $s = 1$ pole of $L(1_B, s)$.

To compute this residue, Davenport and Erdos defined

$$A_k(s) := \frac{1}{a_k^s} - \sum_{j<k} \frac{1}{[a_j, a_k]^s} + \sum_{i<j<k} \frac{1}{[a_i, a_j, a_k]^s} - \dots$$
$$A_0(s) := \sum_{k\geq 1} A_k(s)$$

(so that $A_0(1) = A_0$). Then with these definitions, one can see that

$$L(1_B, s) = \zeta(s) A_0(s),$$

at least where the corresponding Dirichlet series converge. After some additional work, Davenport and Erdős were then able to use this formula for $L(1_B, s)$ to show that the $s = 1$ residue of $L(1_B, s)$ is $A_0(1) = A_0$. This implies that the logarithmic density of $B = \mathbb{N}A$ is A_0, as predicted.

Fourteen years later in 1951 [5], Davenport and Erdős further showed that every set of multiples also has a multiplicative density, equal to its logarithmic density. Their proof in this paper was more direct, proceeding by a careful analysis of the y-smooth elements of B (and estimating their contribution to the corresponding indicator sums).

Putting all of these results together verifies the second row of the table in Theorem 1.1.

Additionally, note that for any set $B \subseteq \mathbb{N}$; $\overline{d}(B) = 1 - \underline{d}(B^c)$, $\overline{\delta}(B) = 1 - \underline{\delta}(B^c)$, and $\overline{\Delta}(B) = 1 - \underline{\Delta}(B^c)$. This means that a set B has a natural/logarithmic/multiplicative density if and only if its complement has a natural/logarithmic/multiplicative density. Hence since saturated sets are precisely the complements of sets of multiples, the above results also imply the third row of the table in Theorem 1.1.

4. Multiplicatively closed saturated sets

Let $B \subseteq \mathbb{N}$ be multiplicatively closed and saturated. It is straightforward to see this means that 1_B is a completely multiplicative function. The indicator 1_B being multiplicative then makes this multiplicative structure the easiest to study by far. Note that the existence of logarithmic and multiplicative densities follows from the previous section. However, one can also see the existence of each type of density directly.

Because 1_B is multiplicative, we have directly from the Halasz mean value theorem [15, 11] that 1_B has a mean value. Hence B has a natural (and also a logarithmic) density.

Next, write

$$\alpha_y = \sum_{n\geq 1} \frac{1}{n} 1_{\mathbb{N}_y}(n) \qquad \text{and} \qquad \beta_y = \sum_{n\geq 1} \frac{1}{n} 1_{\mathbb{N}_y}(n) 1_B(n),$$

so that $\Delta(B) = \lim_{y\to\infty} \frac{\beta_y}{\alpha_y}$. Then since $1_{\mathbb{N}_y}$ and 1_B are both completely multiplicative, observe that

$$\alpha_y = \prod_p \sum_{r\geq 0} \frac{1}{p^r} 1_{\mathbb{N}_y}(p^r) = \prod_{p\leq y} \left(1 - \frac{1}{p}\right)^{-1}$$
$$\text{and} \quad \beta_y = \prod_p \sum_{r\geq 0} \frac{1}{p^r} 1_{\mathbb{N}_y}(p^r) 1_B(p^r) = \prod_{p\leq y, p\in B} \left(1 - \frac{1}{p}\right)^{-1}.$$

Hence $\frac{\beta_y}{\alpha_y}$ is a decreasing function in y (and is bounded below by 0), and so $\Delta(B) = \lim_{y\to\infty} \frac{\beta_y}{\alpha_y}$ necessarily exists.

These arguments verify the third row of the table in Theorem 1.1.

5. Multiplicatively closed sets

In this section, we first show that multiplicatively closed sets are not guaranteed to have a multiplicative density. In fact, we construct an example in the most extreme case; a multiplicatively closed set B with $\underline{\Delta}(B) = 0$ and $\overline{\Delta}(B) = 1$.

Proposition 5.1. *There exists a multiplicatively closed set with lower multiplicative density* 0 *and upper multiplicative density* 1*.*

Proof. For a set of primes P, let $B_P := \dot\bigcup_{p\in P} p\mathbb{N}_p$. Note that $p\mathbb{N}_p$ here is precisely the set of all natural numbers whose largest prime divisor is p. Also note that B_P is multiplicatively closed: $a, b \in B_P$ means that $a \in p\mathbb{N}_p$, $b \in q\mathbb{N}_q$ for $p, q \in P$, which then implies that $ab \in p'\mathbb{N}_{p'} \subseteq B_P$ for $p' = \max(p, q)$.

Now, write

$$\alpha_y = \sum_{n\in\mathbb{N}_y} \frac{1}{n} \qquad \text{and} \qquad \beta_y = \sum_{n\in\mathbb{N}_y} \frac{1}{n} 1_{B_P}(n).$$

Then we will construct P in such a way so that $\underline{\Delta}(B_P) = \liminf_{y\to\infty} \frac{\beta_y}{\alpha_y}$ is 0, and $\overline{\Delta}(B_P) = \limsup_{y\to\infty} \frac{\beta_y}{\alpha_y}$ is 1.

Observe that

$$\alpha_y = \sum_{n\in\mathbb{N}_y} \frac{1}{n} = \prod_{p\le y} \left(1 - \frac{1}{p}\right)^{-1},$$

$$\text{and}\ \ \beta_y = \sum_{n\in\mathbb{N}_y} \frac{1}{n} 1_{B_P}(n) = \sum_{n\in B_P\cap\mathbb{N}_y} \frac{1}{n} = \sum_{p\le y,\, p\in P}\ \sum_{n\in p\mathbb{N}_p} \frac{1}{n}$$

$$= \sum_{p\le y,\, p\in P} \frac{1}{p} \sum_{n\in\mathbb{N}_p} \frac{1}{n} = \sum_{p\le y,\, p\in P} \frac{1}{p}\alpha_p.$$

Then from these identities, it is easy to see that for all primes p,

$$\alpha_p = \left(1 - \frac{1}{p}\right)^{-1} \alpha_{p-1},$$

$$\text{and}\ \ \beta_p = \beta_{p-1} + 1_P(p)\frac{1}{p}\alpha_p.$$

Hence we can compute that

$$\frac{\beta_p}{\alpha_p} - \frac{\beta_{p-1}}{\alpha_{p-1}} = \begin{cases} \dfrac{\beta_{p-1}}{\left(1-\frac{1}{p}\right)^{-1}\alpha_{p-1}} - \dfrac{\beta_{p-1}}{\alpha_{p-1}} = -\dfrac{1}{p}\dfrac{\beta_{p-1}}{\alpha_{p-1}} & \text{if}\ \ p \notin P, \\ \dfrac{\beta_{p-1} + \frac{1}{p}\alpha_p}{\left(1-\frac{1}{p}\right)^{-1}\alpha_{p-1}} - \dfrac{\beta_{p-1}}{\alpha_{p-1}} = \dfrac{1}{p}\left(1 - \dfrac{\beta_{p-1}}{\alpha_{p-1}}\right) & \text{if}\ \ p \in P. \end{cases}$$

We now construct P by processing the primes p in increasing order and choosing whether or not to include each p in P. Note that at each prime p, we will either decrease the quotient $\frac{\beta_y}{\alpha_y}$ by $\frac{1}{p}\frac{\beta_{p-1}}{\alpha_{p-1}}$ (by selecting $p \notin P$), or increase it by $\frac{1}{p}\left(1 - \frac{\beta_{p-1}}{\alpha_{p-1}}\right)$ (by selecting $p \in P$).

We divide this selection process into stages $k = 1, 2, 3, \ldots$.

- At each odd stage k, select primes $p \notin P$ until $\frac{\beta_y}{\alpha_y} \le \frac{1}{k}$.
  - This is possible because while $\frac{\beta_y}{\alpha_y} > \frac{1}{k}$, each choice of $p \notin P$ decreases the quotient $\frac{\beta_y}{\alpha_y}$ by $\frac{1}{p}\frac{\beta_{p-1}}{\alpha_{p-1}} \ge \frac{1}{pk}$ (and $\sum_{p\ge p_0} \frac{1}{pk}$ diverges).
- At each even stage k, select primes $p \in P$ until $\frac{\beta_y}{\alpha_y} \ge 1 - \frac{1}{k}$.
  - This is possible because while $\frac{\beta_y}{\alpha_y} < 1 - \frac{1}{k}$, each choice of $p \in P$ increases the quotient $\frac{\beta_y}{\alpha_y}$ by $\frac{1}{p}\left(1 - \frac{\beta_{p-1}}{\alpha_{p-1}}\right) \ge \frac{1}{pk}$ (and $\sum_{p\ge p_0} \frac{1}{pk}$ diverges).

Hence by construction,

$$\underline{\Delta}(B_P) = \liminf_{y\to\infty} \frac{\beta_y}{\alpha_y} \le \liminf_{\text{odd } k\to\infty} \frac{1}{k} = 0$$
$$\text{and } \ \overline{\Delta}(B_P) = \limsup_{y\to\infty} \frac{\beta_y}{\alpha_y} \ge \limsup_{\text{even } k\to\infty} 1 - \frac{1}{k} = 1,$$

as desired. □

The same multiplicatively closed set from Proposition 5.1 also turns out to not have a logarithmic or natural density. This fact follows from the following lemma.

Lemma 5.2. *Let γ denote Euler's constant. Then $\underline{\delta}(B) \le e^\gamma \underline{\Delta}(B)$ for all $B \subseteq \mathbb{N}$.*

Proof. Write

$$\alpha_y = \sum_{n\in\mathbb{N}_y} \frac{1}{n} = \prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1} \qquad \text{and} \qquad \beta_y = \sum_{n\in\mathbb{N}_y} \frac{1}{n} 1_{B_P}(n),$$

so that $\underline{\Delta}(B) = \liminf_{y\to\infty} \frac{\beta_y}{\alpha_y}$.

We have by Merten's theorem that $\alpha_y \sim e^\gamma \log y$. Hence

$$e^\gamma \underline{\Delta}(B) = \liminf_{y\to\infty} \frac{\beta_y}{\log y} = \liminf_{y\to\infty} \sum_{n\in\mathbb{N}_y} \frac{1}{n} 1_B(y)$$
$$\ge \liminf_{y\to\infty} \sum_{n\le y} \frac{1}{n} 1_B(y) = \underline{\delta}(B),$$

as desired. □

Let $B \subseteq \mathbb{N}$ be the multiplicatively closed set from Proposition 5.1. Then applying Lemma 5.2, we obtain that

$$\underline{d}(B) \le \underline{\delta}(B) \le e^\gamma \underline{\Delta}(B) = 0$$
$$\text{and } \ \overline{d}(B) \ge \overline{\delta}(B) = 1 - \underline{\delta}(B^c) \ge 1 - e^\gamma \underline{\Delta}(B^c) = 1.$$

Thus B does not have a multiplicative, logarithmic, or natural density. This verifies the first row of the table in Theorem 1.1.

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(E. Ross) School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC

Email address: `erickjohnross@gmail.com`