HECKE EIGENVALUE EQUIDISTRIBUTION OVER THE NEWSPACES WITH NEBENTYPUS

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ABSTRACT. Fix a prime p, and let $\widehat{T}_p^{\mathrm{new}}(N,k,\chi) := \chi(p)^{-1/2}p^{-(k-1)/2}T_p^{\mathrm{new}}(N,k,\chi)$ denote the normalized p'th Hecke operator over the newspace with nenbentypus $S_k^{\mathrm{new}}(N,\chi)$. In this paper, we determine the distribution of the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$ as $N+k\to\infty$.

1. Introduction

For $N \geq 1$ and $k \geq 2$ even, let $S_k(N)$ denote the space of cuspidal modular forms of weight k and modular group $\Gamma_0(N)$. Additionally, let $S_k^{\mathrm{new}}(N) \subseteq S_k(N)$ denote it's new subspace. For primes p, let $\widehat{T}_p(N,k) := p^{-(k-1)/2}\,T_p(N,k)$ and $\widehat{T}_p^{\mathrm{new}}(N,k) := p^{-(k-1)/2}\widehat{T}_p^{\mathrm{new}}(N,k)$ denote the p'th normalized Hecke operators over $S_k(N)$ and $S_k^{\mathrm{new}}(N)$, respectively.

The asymptotic behavior of the eigenvalues of these Hecke operators has been studied in many different settings. From one asymptotic perspective, one can fix N and k, and ask about the behavior of the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k)$ as $p\to\infty$. Here, the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k)$ correspond with the normalized Fourier coefficients $\widehat{a}_f(p)$ of newforms $f = \sum_{m\geq 1} m^{(k-1)/2}\widehat{a}_f(m)q^m \in S_k^{\mathrm{new}}(N)$. And the asymptotic behavior of these Fourier coefficients has been an important area of research in Number Theory for many years. The Sato-Tate conjecture (now a theorem [1]), for example, states that the prime-indexed Fourier coefficients $\widehat{a}_f(p)$ of a non-CM newform f tend to the Sato-Tate distribution μ_∞ as $p\to\infty$. Similarly, it is known that the prime-indexed Fourier coefficients $\widehat{a}_f(p)$ of a CM newform tend to the CM distribution μ_{CM} as $p\to\infty$ [2, Theorem 15.4].

From a slightly different perspective, one can fix p and ask about the behavior of the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k)$ as $N+k\to\infty$. From this perspective, Serre showed that the eigenvalues of $\widehat{T}_p(N,k)$ and $\widehat{T}_p^{\mathrm{new}}(N,k)$ tend to yet another distribution μ_p as $N+k\to\infty$ [6, Theorems 1, 2].

A natural question one might then ask is if the same result holds over the spaces with nebentypus $S_k(N,\chi)$ and $S_k^{\mathrm{new}}(N,\chi)$. In fact, Serre noted that his result (with the same distribution μ_p) could be extended to the full spaces with nebentypus $S_k(N,\chi)$ via an identical strategy [6, Theorem 4]. However, [6] did not contain the tools to show the corresponding result over the newspaces with nebentypus $S_k^{\mathrm{new}}(N,\chi)$. In this paper, we address this case by determining the distribution of the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$.

2020 *Mathematics Subject Classification.* Primary 11F11; Secondary 11F25, 11F72.

Key words and phrases. Modular forms; Hecke operators; Newspace; Nebentypus; Equidistribution.

One might guess that Serre's μ_p-equidistribution result should still hold over $S_k^{\mathrm{new}}(N,\chi)$. However, this turns out to not be true; the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$ are not guaranteed to tend to a distribution (albeit for a trivial reason explained shortly). But when these eigenvalues do tend to an asymptotic distribution, they will tend to μ_p, like for the full space $S_k(N,\chi)$.

In particular, we show the following result. Here, and throughout this paper, $\mathfrak{f}(\chi)$ denotes the conductor of χ.

Theorem 1.1. *Fix a prime p, and let $\mu_p(x) := \frac{p+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2-x^2}dx$. Consider $N \geq 1$ coprime to p, $k \geq 2$, and χ Dirichlet characters modulo N such that $\chi(-1) = (-1)^k$. Then assuming it is not the case that $2 \mid \mathfrak{f}(\chi)$, $2 \parallel N/\mathfrak{f}(\chi)$; the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$ are μ_p-equidistributed as $N+k \to \infty$.*

We note that when $2 \mid \mathfrak{f}(\chi)$, $2 \parallel N/\mathfrak{f}(\chi)$, the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$ do not tend to a distribution for a trivial reason. In this case, it turns out that $\dim S_k^{\mathrm{new}}(N,k,\chi) = 0$ [5, Proposition 6.1], so in fact, there are no eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$. However, if we exclude the case of $2 \mid \mathfrak{f}(\chi)$, $2 \parallel N/\mathfrak{f}(\chi)$, then it turns out that $\dim S_k^{\mathrm{new}}(N,k,\chi) \to \infty$ as $N+k \to \infty$ [5, Theorem 1.3]. Hence after excluding this exceptional case, it is perfectly well-defined to ask about the distribution of the eigenvalues of $\widehat{T}_p(N,k,\chi)$ as $N+k \to \infty$.

Finally, in Section 3, we will discuss an application of Theorem 1.1. In particular, one can use the equidistribution result from Theorem 1.1 to obtain information about the coefficient fields for newforms in $S_k^{\mathrm{new}}(N,\chi)$.

Most of the technical details underlying the techniques used in this paper have already been worked out in other contexts (both by the author, and by others). In particular, we rely heavily on two key pieces. First, we use a certain characterization of μ_p-equidistribution from [6] (following the same outline that Serre used to show his original μ_p-equidistribution result). Second, we utilize the trace estimates computed in [5] and [3].

2. Proof of the Main Theorem

In this section, we prove the main theorem. Define $\widehat{T}_m^{\mathrm{new}}(N,k,\chi)$ via the normalization $\widehat{T}_m^{\mathrm{new}}(N,k) := \chi(m)^{-1/2}m^{-(k-1)/2}\,T_m(N,k,\chi)$ (taking, for example, the principal branch of $\chi(p)^{-1/2}$ for primes p). We are using this normalization so that the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$ all lie in the real interval $[-2,2]$.

To show the μ_p-equidistribution of the eigenvalues of $\widehat{T}_p^{\mathrm{new}}(N,k,\chi)$, it suffices [6, Proposition 2] to verify the following identity for the Chebyshev polynomials $X_n(x) := U_n(x/2)$:

$$\frac{1}{\dim S_k^{\mathrm{new}}(N,\chi)} \sum_{\lambda \in \mathrm{eigv}\,\widehat{T}_p(N,k,\chi)} X_n(\lambda) \longrightarrow \int_{-2}^{2} X_n(x)\mu_p(x) \quad \text{as} \quad N+k \to \infty. \tag{$*$}$$

Just showing this identity for the Chebyshev polynomials turns out to be sufficient because the Chebyshev polynomials form a basis for the space of all polynomials, and the polynomials are dense in the space of all continuous functions.

First, we evaluate the RHS of ($*$). The value of this integral was already computed in [6, Equation (20)], but we repeat a proof here for convenience of the reader.

Lemma 2.1. *For each $n \geq 0$,*

$$\int_{-2}^{2} X_n(x)\mu_p(x) = \mathbb{1}_{2|n}\, p^{-n/2}.$$

Proof. We make use of the two distributions

$$\mu_\infty(x) := \frac{1}{\pi}\left(1 - x^2/4\right)^{1/2} dt \qquad \text{and} \qquad \mu_p(x) := \frac{p+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2 - x^2}dx.$$

Now, recall that the generating function for the $X_k(x)$ is given by

$$\sum_{k=0}^{\infty} X_k(x)t^k = \frac{1}{1 - tx + t^2}.$$

Substituting $t = p^{-1/2}$ and $t = -p^{-1/2}$ into this identity and summing the two resulting equations then yields

$$\begin{aligned}\sum_{k=0}^{\infty} X_k(x)\mathbb{1}_{2|k}2p^{-k/2} &= \frac{1}{1 - p^{-1/2}x + p^{-1}} + \frac{1}{1 + p^{-1/2}x + p^{-1}} \\ &= \frac{2 + 2p^{-1}}{(1+p^{-1})^2 - p^{-1}x^2} = 2\frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2} = 2\frac{\mu_p(x)}{\mu_\infty(x)}.\end{aligned}$$

Thus

$$\begin{aligned}\int_{-2}^{2} X_n(x)\mu_p(x) &= \int_{-2}^{2} X_n(x)\sum_{k=0}^{\infty} X_k(x)\mathbb{1}_{2|k}p^{-k/2}\mu_\infty(x) \\ &= \sum_{k=0}^{\infty}\mathbb{1}_{2|k}p^{-k/2}\int_{-2}^{2} X_n(x)X_k(x)\mu_\infty(x) = \mathbb{1}_{2|n}p^{-n/2},\end{aligned}$$

as desired. Note that in the last step, we used the well-known fact that the Chebyshev polynomials are μ_∞-orthonormal (i.e. that $\int_{-2}^{2} X_n(x)X_k(x)\,\mu_\infty(x) = \delta_{n,k}$). □

Next, we evaluate the LHS of ($*$). Observe that the classical recurrence relation for the $\widehat{T}_{p^n}$ [4, Theorem 10.2.9] is identical to the defining recurrence relation for the Chebyshev polynomials:

$$\begin{aligned}\widehat{T}_{p^0} &= \mathrm{Id}, & X_0(x) &= 1, \\ \widehat{T}_{p^1} &= \widehat{T}_p, & X_1(x) &= x, \\ \widehat{T}_{p^{n+1}} &= \widehat{T}_p\,\widehat{T}_{p^n} - \widehat{T}_{p^{n-1}}, & X_{n+1}(x) &= x\,X_n(x) - X_{n-1}(x).\end{aligned}$$

This means that the LHS of (∗) is equal to

$$\begin{aligned}\text{LHS} &= \frac{1}{\dim S_k^{\text{new}}(N,\chi)} \sum_{\lambda\in \text{eigv}\, \widehat{T}_p(N,k,\chi)} X_n(\lambda)\\ &= \frac{1}{\dim S_k^{\text{new}}(N,\chi)} \sum_{\lambda\in \text{eigv}\, \widehat{T}_{p^n}(N,k,\chi)} \lambda\\ &= \frac{\operatorname{Tr} \widehat{T}_{p^n}^{\text{new}}(N,k,\chi)}{\operatorname{Tr} \widehat{T}_1^{\text{new}}(N,k,\chi)}.\end{aligned}$$

Hence, to verify the identity (∗), it suffices to show that

$$\frac{\operatorname{Tr} \widehat{T}_{p^n}^{\text{new}}(N,k,\chi)}{\operatorname{Tr} \widehat{T}_1^{\text{new}}(N,k,\chi)} \longrightarrow \mathbb{1}_{2|n} p^{-n/2} \quad \text{as} \quad N+k\to\infty. \tag{$**$}$$

Asymptotic estimates on these traces were recently computed in [5] (estimating the growth of the main term) and [3] (bounding the error terms). In particular, we have from [3, Lemma 6.2] that

$$\operatorname{Tr} \widehat{T}_m^{\text{new}}(N,k,\chi) = \frac{\mathbb{1}_{m=\square}}{\sqrt{m}} \frac{k-1}{12} \psi_{\mathfrak{f}(\chi)}^{\text{new}}(N) + O(N^{1/2+\varepsilon}), \tag{2.1}$$

(where the big-O notation here is with respect to both N and k). If the reader is interested, the exact definition of $\psi_{\mathfrak{f}(\chi)}^{\text{new}}(N)$ can be found in [3, Equation 6.3]. However, for our purposes, we will only use the lower bound for $\psi_{\mathfrak{f}(\chi)}^{\text{new}}(N)$ computed in [5, Equation 6.2]:

$$\psi_{\mathfrak{f}(\chi)}^{\text{new}}(N) \geq \frac{N}{\nu(N)} \qquad \text{where} \qquad \nu(N) := \prod_{p|N} \begin{cases} 4 & \text{if } p=2\\ \left(1+\frac{2}{p-2}\right) & \text{if } p\neq 2.\end{cases}$$

Note that in particular, $\nu(N) \leq 4^{\omega(N)} = O(N^{\varepsilon})$, so $\psi_{\mathfrak{f}(\chi)}^{\text{new}}(N) \geq N/\nu(N) = \Omega(N^{1-\varepsilon})$.

Then using the estimate (2.1) at $m=1$ and $m=p^n$, we obtain that

$$\frac{\operatorname{Tr} \widehat{T}_{p^n}^{\text{new}}(N,k,\chi)}{\operatorname{Tr} \widehat{T}_1^{\text{new}}(N,k,\chi)} = \frac{\mathbb{1}_{2|n} p^{-n/2} \frac{k-1}{12} \psi_{\mathfrak{f}(\chi)}^{\text{new}}(N) + O(N^{1/2+\varepsilon})}{\frac{k-1}{12} \psi_{\mathfrak{f}(\chi)}^{\text{new}}(N) + O(N^{1/2+\varepsilon})} \longrightarrow \mathbb{1}_{2|n} p^{-n/2} \quad \text{as} \quad N+k\to\infty.$$

This verifies (∗∗), completing the proof of Theorem 1.1.

3. An Application of the Main Theorem

Finally, we point out one application of Theorem 1.1, analogous to [6, Theorem 5].

For newforms $f \in S_k^{\text{new}}(N,\chi)$, let $K_f := \mathbb{Q}(\{a_f(m) : m \geq 1,\ (m,N)=1\})$ denote the coefficient field of f. Then for $r \geq 1$, let $s(N,k,\chi)_r$ denote the number of newforms $f \in S_k^{\text{new}}(N,\chi)$ such that $[K_f : \mathbb{Q}] = r$. Then the equidistribution result of Theorem 1.1 implies the following corollary.

Corollary 3.1. *Fix p prime, $k \geq 2$, and $r \geq 1$. Consider N coprime to p and Dirichlet characters χ modulo N with $\chi(-1) = (-1)^k$. Then assuming it is not the case that $2 \mid \mathfrak{f}(\chi)$, $2 \parallel N/\mathfrak{f}(\chi)$;*

$$\frac{s(N,k,\chi)_r}{\dim S_k^{\text{new}}(N,k,\chi)} \longrightarrow 0 \qquad \textit{as} \qquad N \longrightarrow \infty.$$

Recall that the Fourier coefficients $a_f(m)$ correspond with the eigenvalues of T_m. Then the above corollary follows from the fact that the set of possible T_m eigenvalues with degree $\leq r$ is finite, so the set of normalized eigenvalues is μ_p-measure 0 in $[-2, 2]$. The interested reader can find the precise details of this proof in [6, Theorem 5].

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